# Chapter 3 Optimization



Key-words

- Lagrange multiplier
- Eigenvalue and eigenvector
- Characteristic polynomial
- SD(Spectral Decomposition) and SVD(Singular Value Decomposition)

# 3.1 Dimensional reduction of the multivariate data

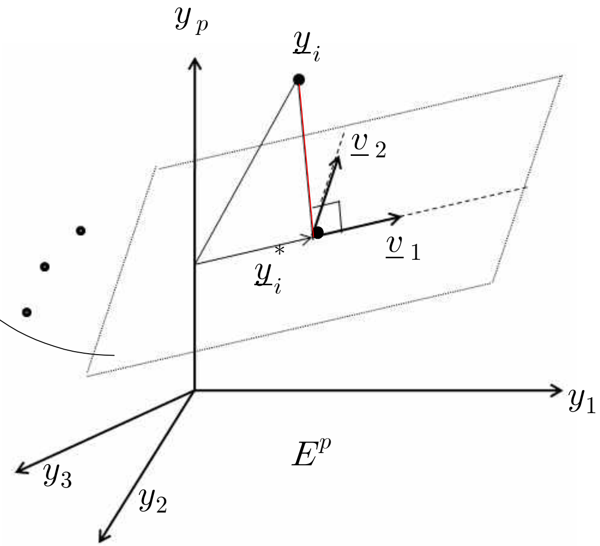
$$X_{n\times p}=(x_{ij}) \quad \Rightarrow \quad Y=(y_{ij})=(x_{ij}-\overline{x}_j)=\begin{pmatrix} \underline{y}_1^t \\ \vdots \\ \underline{y}_i^t \\ \vdots \\ \underline{y}_n^t \end{pmatrix} \quad , \quad \underline{y}_i^t=(x_{i1}-\overline{x}_1, \; \cdots \; , x_{ip}-\overline{x}_p)$$

$$\Leftrightarrow \quad \underline{y}_i=\begin{pmatrix} x_{i1}-\overline{x_1} \\ \vdots \\ x_{ip}-\overline{x_p} \end{pmatrix}=\begin{pmatrix} y_{i1} \\ \vdots \\ y_{ip} \end{pmatrix} \; , \; i=1,2,\cdots,n \qquad \in E^p$$

$\exists \, E^s=span(\underline{v}_1,\underline{v}_2,\cdots,\underline{v}_s)$ and $\underline{y}_i^*$: projection of n points $\underline{y}_i$ onto the vectors $\underline{v}_1,\ldots,\underline{v}_s \in E^s$

$$\Rightarrow \min \sum_{i=1}^{n} \| \underline{y}_i-\underline{y}_i^* \|^2$$

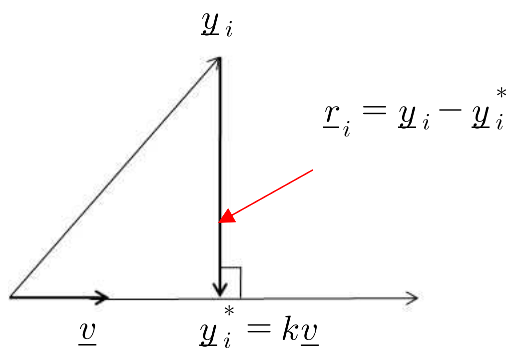
with $\quad \underline{v}_j^t\underline{v}_k=0, \; \underline{v}_j^t\underline{v}_j=1 \equiv \| \underline{v}_j \|$



**Projection:** Projection $\underline{y}_i^*$ of the $n$ points $\underline{y}_i,\ i=1,\dots,n$ onto the vector $\underline{v}$ .

$$\underline{y}_i^* = k\underline{v}_{p\times 1},$$

$\underline{r}_i = \underline{y}_i - \underline{y}_i^*$ : residual vector




Projection of $\underline{y}_i$ onto a one-diemensional subspace

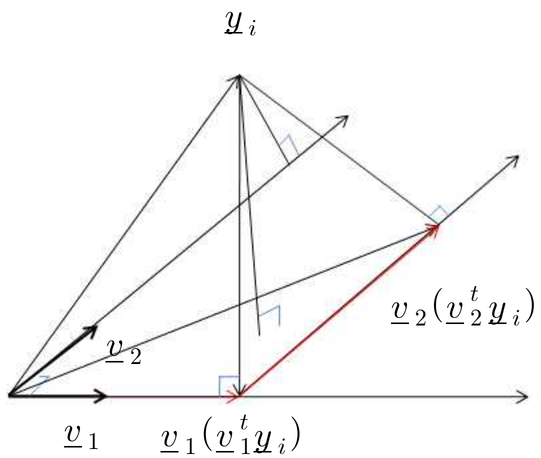
$$\underline{r}_i \perp \underline{v} \iff \underline{r}_i \bullet \underline{v} = 0$$

$$\iff (\underline{y}_i - k\underline{v})^t \underline{v} = 0$$

$$\iff k = \frac{\underline{y}_i^t \underline{v}}{\underline{v}^t \underline{v}} = \underline{y}_i^t \underline{v} \ (\because \|\underline{v}\| = 1)$$

$$= \underline{v}^t \underline{y}_i$$

$$\therefore\ \underline{y}_i^* \equiv \underline{v}\,(\underline{v}^t \underline{y}_i)$$




Projection of $\underline{y}_i$ onto a $s$−diemensional subspace

$$\therefore\ \underline{y}_i^* = \underline{v}_1(\underline{v}_1^t\underline{y}_i) + \underline{v}_2(\underline{v}_2^t\underline{y}_i) + \cdots + \underline{v}_s(\underline{v}_s^t\underline{y}_i) = \sum_{k=1}^{s}\underline{v}_k\underline{v}_k^t\underline{y}_i$$

$$= VV^t\underline{y}_i \qquad \text{where } V = (\underline{v}_1, \cdots, \underline{v}_s)$$

$$\underline{y}_i - \underline{y}_i^* = \underline{y}_i - VV^t\underline{y}_i = (I_p - VV^t)\underline{y}_i$$

$$\| \underline{y}_i - \underline{y}_i^* \|^2 = (\underline{y}_i - \underline{y}_i^*)^t(\underline{y}_i - \underline{y}_i^*)$$

$$= \underline{y}_i^t(I_p - VV^t)(I_p - VV^t)\underline{y}_i$$

$$= \underline{y}_i^t(I_p - VV^t)\underline{y}_i \quad \left(\because (I_p - VV)^t(I_p - VV^t) = (I_p - VV^t)\right)$$

$$\therefore \quad S = \sum_{i=1}^{n} \underline{y}_i^t (I_p - VV^t) \underline{y}_i$$

$$= \sum_{i=1}^{n} \left[ \underline{y}_i^t \underline{y}_i - \underline{y}_i^t V V^t \underline{y}_i \right]$$

$$= \sum_{i=1}^{n} tr \left[ \underline{y}_i^t \underline{y} - \underline{y}_i^t V V^t \underline{y}_i \right]$$

$$= \sum_{i=1}^{n} tr \left[ \underline{y}_i \underline{y}_i^t - V^t \underline{y}_i \underline{y}_i^t V \right]$$

$$= tr \left[ \sum_{i=1}^{n} \underline{y}_i \underline{y}_i^t - V^t \sum_{i=1}^{n} \underline{y}_i \underline{y}_i^t V \right]$$

$$= tr \left[ Y^t Y - V^t Y^t Y V \right]$$

$$= \sum_{i=1}^{n} \underline{y}_i^t \underline{y}_i - \sum_{k=1}^{s} \underline{v}_k^t Y^t Y \underline{v}_k$$

$$\therefore \quad S = \sum_{i=1}^{n} \underline{y}_i^t \underline{y}_i - \sum_{k=1}^{s} \underline{v}_k^t Y^t Y \underline{v}_k$$

$$\min_{\underline{v}_k} S \quad \Leftrightarrow \quad \max_{\underline{v}_k} \sum_{k=1}^{s} \underline{v}_k^t Y^t Y \underline{v}_k$$

## ■ Optimal problem in dimensional reduction of multivariate data

$\underline{y}_i^* = \underline{v}_k(\underline{v}_k^t \underline{y}_i)$ : Projection of the $n$ points $\underline{y}_i$, $i=1,\ldots,n$

onto the basis vector $\underline{v}_k$ of $k$th dimensional subspace.

All at once, it is impossible to obtain all $\underline{v}_k$, $k=1,\cdots,p$.

Subsequently, it is possible to find $\underline{v}_k$.

Therefore consider the optimal problem of maximizing $\underline{v}^t Y^t Y \underline{v}$ for any $\underline{v}$ s.t. $\underline{v}^t \underline{v}=1$.

This optimal problem in dimensional reduction of multivariate data can be solved by

Lagrange multipliers method

$$f(\underline{v},\lambda) = \underline{v}^t Y^t Y \underline{v} - \lambda(\underline{v}^t \underline{v} - 1)$$

$$\frac{\partial f}{\partial \underline{v}} = 2Y^t Y \underline{v} - 2\lambda \underline{v} = 0 \quad \Leftrightarrow \quad Y^t Y \underline{v} = \lambda \underline{v}$$ : Eigenvalue-eigenvector equation

# 3.2 Eigenvalue and eigenvector

## ■ Definition of eigenvalue and eigenvector

Given a $A_{p\times p}$,

eigenvalue–eigenvector equation: $A\underline{v} = \lambda\underline{v}$ $\Rightarrow$ $\lambda$ : eigenvalues

$\underline{v}(\neq\underline{0})$ : eigenvector corresponding to $\lambda$

Note

$$A\underline{v} - \lambda\underline{v} = \underline{0} \quad \Leftrightarrow \quad (A-\lambda I)\underline{v} = \underline{0}$$

- $A-\lambda I$ : non–singular $\Leftrightarrow$ $|A-\lambda I| \neq 0$ $\Leftrightarrow$ $\exists\,(A-\lambda I)^{-1}$ $\Rightarrow$ $\underline{v} = \underline{0}$(contradiction!)
- $A-\lambda I$ : singular $\Rightarrow$ $\underline{v} = [(A-\lambda I)^{-}(A-\lambda I) - I]\underline{z}$ for arbitrary $\underline{z}$

Result

$\boxed{\exists\ \lambda\text{(eigenvalue)} \ \Rightarrow\ |A-\lambda I| = 0 \ :\ \text{Characteristic(determinantal) equation of } A}$

Example 1

$$A=\begin{pmatrix}1 & 1\\ -2 & 4\end{pmatrix} \Rightarrow \begin{pmatrix}1 & 1\\ -2 & 4\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\lambda\begin{pmatrix}v_1\\ v_2\end{pmatrix} \Leftrightarrow \begin{pmatrix}1-\lambda & 1\\ -2 & 4-\lambda\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{vmatrix}1-\lambda & 1\\ -2 & 4-\lambda\end{vmatrix}=0 \Leftrightarrow (l-\lambda)(4-\lambda)+2=4-5\lambda+\lambda^2+2$$

$$=\lambda^2-5\lambda+6$$

$$=(\lambda-3)(\lambda-2)=0$$

$$\Leftrightarrow \lambda=3 \text{ or } \lambda=2$$

i ) $\lambda=2 \Rightarrow \begin{pmatrix}-1 & 1\\ -2 & 2\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Leftrightarrow -v_1+v_2=0 \Leftrightarrow v_1=v_2$: arbitrary

$\therefore$ $\underline{v}=k(1,1)^t$ , $k$: constant

ii ) $\lambda=3 \Rightarrow \begin{pmatrix}-2 & 1\\ -2 & 1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Leftrightarrow -2v_1+v_2=0 \Leftrightarrow v_2=2v_1$

$\therefore$ $\underline{v}=k(1,2)^t$ , $k$: constant

Example 2: complex eigenvalues and eigenvectors

$$A=\begin{pmatrix}1 & 1\\ -2 & -1\end{pmatrix}$$

$$\begin{vmatrix}1-\lambda & 1\\ -2 & -1-\lambda\end{vmatrix} = -(1-\lambda)(1+\lambda)+2$$

$$=\lambda^2+1=0$$

$$\therefore\ \lambda=\pm\sqrt{-1}=\pm i$$

i ) $\lambda=i \Rightarrow \begin{pmatrix}1 & 1\\ -2 & -1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=i\begin{pmatrix}v_1\\ v_2\end{pmatrix} \Leftrightarrow \begin{cases}v_1+v_2=i\,v_1\\ -2v_1-v_2=i\,v_2\end{cases} \Leftrightarrow -2\,v_1=(1+i)v_2$

$$\Leftrightarrow v_1=\frac{-(1+i)v_2}{2},\ v_2=-2k \quad\Rightarrow\quad \underline{v}=k\begin{pmatrix}1+i\\ -2\end{pmatrix},\ k\neq 0$$

ii ) $\lambda=-i \Rightarrow \begin{pmatrix}1 & 1\\ -2 & -1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=-i\begin{pmatrix}v_1\\ v_2\end{pmatrix} \Leftrightarrow \begin{cases}v_1+v_2=-i\,v_1\\ -2v_1-v_2=-i\,v_2\end{cases} \Leftrightarrow -2v_1=(1-i)v_2$

$$\Leftrightarrow v_1=\frac{-(1-i)v_2}{2},\ v_2=-2k \quad\Rightarrow\quad \underline{v}=k\begin{pmatrix}1-i\\ -2\end{pmatrix},\ k\neq 0$$

Example 3

$$A=\begin{pmatrix}2 & 1 & 1\\ 2 & 3 & 4\\ -1 & -1 & -2\end{pmatrix}$$

$$\begin{vmatrix}2-\lambda & 1 & 1\\ 2 & 3-\lambda & 4\\ -1 & -1 & -2-\lambda\end{vmatrix}=(2-\lambda)(3-\lambda)(-2-\lambda)-2-4-\{-(3-\lambda)+2(-2-\lambda)-4(2-\lambda)\}$$

$$=(\lambda+1)(\lambda-1)(\lambda-3)=0$$

$\therefore\ \lambda=-1$ or $\lambda=1$ or $\lambda=3$

i ) $\lambda=-1 \quad \Rightarrow \quad \begin{pmatrix}3 & 1 & 1\\ 2 & 4 & 4\\ -1 & -1 & -1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}=\underline{0} \quad \Leftrightarrow \quad \begin{cases}3v_1+v_2+v_3=0\\ 2v_1+4v_2+4v_3=0\\ -v_1-v_2-v_3=0\end{cases} \quad \Rightarrow \quad \begin{matrix}v_1=0\\ v_2=-v_3\\ v_3=-v_2\end{matrix}$

$$\Leftrightarrow\ v_1=0,\ v_2=k,\ v_3=-k \quad \Rightarrow \quad \underline{v}=k\begin{pmatrix}0\\ 1\\ -1\end{pmatrix}$$

ii) $\lambda = 1 \quad \Rightarrow \quad \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 4 \\ -1 & -1 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \underline{0} \quad \Leftrightarrow \quad \begin{cases} v_1 + v_2 + v_3 = 0 \\ v_1 + v_2 + 2v_3 = 0 \\ v_1 + v_2 + 3v_3 = 0 \end{cases} \quad \Rightarrow \quad \begin{matrix} v_1 = -v_2 \\ v_2 = -v_1 \\ v_3 = 0 \end{matrix}$

$\Leftrightarrow v_1 = k,\ v_2 = -k,\ v_3 = 0 \quad \Rightarrow \quad \underline{v} = k\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

iii) $\lambda = 3 \quad \Rightarrow \quad \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 4 \\ -1 & -1 & -5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \underline{0} \quad \Leftrightarrow \quad \begin{cases} v_1 - v_2 - v_3 = 0 \\ v_1 + 2v_3 = 0 \\ v_1 + v_2 + 5v_3 = 0 \end{cases} \quad \Rightarrow \quad \begin{matrix} v_1 = -2v_3 \\ v_2 = -3v_3 \\ v_3 = v_1 - v_2 = -\frac{1}{3}v_2 \end{matrix}$

$\Leftrightarrow v_1 = -2k,\ v_2 = -3k,\ v_3 = k \quad \Rightarrow \quad \underline{v} = k\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$

Example 4

$$A=\begin{pmatrix}1&2&3\\0&1&0\\0&2&1\end{pmatrix}$$

$$\begin{vmatrix}1-\lambda&2&3\\0&1-\lambda&0\\0&2&1-\lambda\end{vmatrix}=(1-\lambda)^3=0$$

$$\therefore\ \lambda=1\quad\Rightarrow\quad\begin{pmatrix}0&2&3\\0&0&0\\0&2&0\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=0\quad\Leftrightarrow\quad\begin{cases}2v_2+3v_3=0\\2v_2=0\end{cases}\quad\Rightarrow\quad\begin{matrix}v_2=0\\v_3=0\end{matrix}$$

$$\Rightarrow\quad\underline{v}=k\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\Rightarrow\quad\left\{\begin{pmatrix}k\\0\\0\end{pmatrix}:\ |k|<\infty\right\}=S_A(1)\ :\ \text{Eigenspace}$$

Note : Eigenspace of $A$ associated with $\lambda$

$$S_A(\lambda) \equiv \left\{ \underline{x} \;:\; \underline{x} \in R^m \wedge A\underline{x} = \lambda \underline{x} \right\}$$

: Collection of all eigenvectors corresponding to the particular eigenvalue $\lambda$ along with the trivial vector $\underline{0}$

Remark: Eigenvalue–eigenvector equation of $A^{-1}$

$A$ : non-singular, $A\underline{v} = \lambda \underline{v}$

$$\exists A^{-1} \;\Rightarrow\; A^{-1}A\underline{v} = \lambda A^{-1}\underline{v} \;\Leftrightarrow\; \underline{v} = \lambda A^{-1}\underline{v} \;\Leftrightarrow\; A^{-1}\underline{v} = \lambda^{-1}\underline{v}$$

$\Rightarrow$ $\lambda^{-1}$ : eigenvalue of $A^{-1}$ corresponding to the eigenvector $\underline{v}$

# 3.3 Spectral decomposition

■ Spectral Decomposition (SD) of a symmetric matrix

Given a $p \times p$ symmetric matrix $A$ with eigenvalues $\lambda_{1,} \cdots, \lambda_p$ and

$\{\underline{v}_1, \cdots, \underline{v}_p\}$ is a set of orthonormal eigenvectors corresponding to these eigenvalues

$\Rightarrow \quad A = VD_\lambda V^t$

where $D_\lambda = diag(\lambda_1, \cdots, \lambda_p)$ and $V = (\underline{v}_1, \cdots, \underline{v}_p)$ st $V^t V = VV^t = I.$

Note

$$\text{Orthonomal} \quad \Leftrightarrow \quad \begin{cases} \| \underline{v}_j \| = 1 \\ \underline{v}_j^t \underline{v}_k = 0 \end{cases}$$

## Corollary

i ) $A^{1/2}$ : square root matrix of $A$ satisfying $A = A^{1/2}A^{1/2} > 0$

- $D_{\sqrt{\lambda}} = diag(\lambda_1^{1/2}, \cdots, \lambda_p^{1/2})$
- $A^{1/2} = VD_{\sqrt{\lambda}}V^t$ : symmetric square root of $A$

ii ) $(A^{1/2})^t = (VD_{\sqrt{\lambda}}V^t)^t = VD_{\sqrt{\lambda}}V^t = A^{1/2}$

## Remark

$Y = (x_{ij} - \overline{x}_j)$

- $Y = HX$ where $H = I - \dfrac{1}{n}J$ : centering matrix
- $Y^tY = (n-1)S$ where $S$ : covariance matrix
- $A = Y^tY = VD_{\lambda}V^t = (n-1)S \iff S = VD_{\lambda/(n-1)}V^t$

## ■ SVD (Singular Value Decomposition)

Given a $n \times p$ matrix $Y$,

$$\underset{n\times p}{Y} = \underset{n\times p}{U}\ \underset{p\times p}{D_{\sqrt{\lambda}}}\ \underset{p\times p}{V^t}$$

where $D_{\sqrt{\lambda}} = diag(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_p})$ with $\lambda_1 \geq \cdots \geq \lambda_p$ : eigenvalues of $Y^t Y (= V D_\lambda V^t)$

$$V^t V = V V^t = I\ , \qquad U^t U = I$$

Note: SVD is a factorazation for a matrix of any size.
SD is a factorization for a square and symmetric matrix.

Note : SVD of a $n \times p$ centred data matrix : $Y = (x_{ij} - \bar{x}_{j}) = U D_{\sqrt{\lambda}} V^t$

- $$\begin{aligned} Y^t Y_{p\times p} &= (U D_{\sqrt{\lambda}} V^t)^t (U D_{\sqrt{\lambda}} V^t) \\ &= V D_{\sqrt{\lambda}} U^t U D_{\sqrt{\lambda}} V^t \\ &= V D_\lambda V^t \quad : \text{SD} \quad (= (n-1) S) \end{aligned}$$

- $YY^t = (UD_{\sqrt{\lambda}}V^t)(UD_{\sqrt{\lambda}}V^t)^t$

$$= UD_{\sqrt{\lambda}}V^tVD_{\sqrt{\lambda}}U^t$$

$$= UD_{\lambda}U^t$$ : SD $(\neq (n-1)S)$

Applications in multivariate statistics

- PCA (Principal Component Analysis)
- FA (Factor Analysis)
- MDS (MultiDimensional Scaling)

etc.

■ GSVD (Generalized Singular Value Decomposition)

$$\underset{n\times p}{A} = \underset{n\times r}{\widetilde{U}} \, D_{\sqrt{\lambda}} \, \underset{r\times p}{\widetilde{X}^t}$$

where $\widetilde{U}^t \Omega \widetilde{U} = \widetilde{V}^t \Phi \widetilde{V} = I$ , $D_{\sqrt{\lambda}} = diag(\tilde{\lambda}_1^{1/2}, \cdots, \tilde{\lambda}_r^{1/2})$ with $\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_r \geq 0$

$\widetilde{U}$ : orthonormal w.r.t. $\Omega > 0$

$\widetilde{V}$ : orthonormal w.r.t. $\Phi > 0$

Note

$$\widetilde{U}^t \Omega^{1/2} \Omega^{1/2} \widetilde{U} = U^t U$$

$$\widetilde{V}^t \Phi^{1/2} \Phi^{1/2} \widetilde{V} = V^t V$$

$$\Rightarrow \quad \Omega^{1/2} A \Phi^{1/2} = (\Omega^{1/2} \widetilde{U}) D_{\sqrt{\lambda}} \widetilde{V}^t \Phi^{1/2} = U D_{\sqrt{\lambda}} V^t$$

Application Correspondence Analysis

# 3.4 Matrix decomposition in R and SAS/IML

| $A=VD_\lambda V^t$: SD and $Z=UD_qV^t$: SVD | |
|---|---|
| R | SAS/IML |
| • eigen(A)<br>eigen.A <- eigen(A)<br>round(eigen.A$values, 4)<br>round(eigen.A$vcetors, 4) | • call eigen(lambda, V, A)<br>B=V*diag(lambda)*V'; |
| • svd(Z)<br>svd.Z <- svd(Z)<br>round(svd.Z$d, 4)<br>round(svd.Z$u, 4)<br>round(svd.Z$v, 4) | • call svd(U,q,V,Z)<br>B=U*diabg(q)*V'; |

Refer to the pp.69~71 in your text.

# 3.5 Geometric concept of positive definite symmetric matrix

$\underline{x} \sim N_p(\underline{\mu}, \Sigma), \ \Sigma > 0$

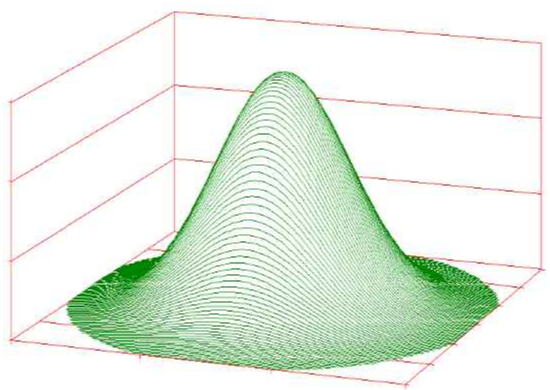
$$f(\underline{x}) = (2\pi)^{-p/2} |\Sigma|^{-1/2} \exp\left(-\frac{1}{2}(\underline{x}-\underline{\mu})^t \Sigma^{-1} (\underline{x}-\underline{\mu}\right)$$

$p = 2$ : Bivariate normal distribution

$$N_2(\underline{\mu}, \Sigma), \ \underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \ \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

let $(\underline{x}-\underline{\mu})^t \Sigma (\underline{x}-\underline{\mu}) = 1$

$\Rightarrow \quad \underline{\mu} = 0, \ \underline{x}^t \Sigma^{-1} \underline{x} = 1$



Note

i ) $\Sigma = VD_{\lambda}V^t$

ii ) $\Sigma^{-1} = VD_{1/\lambda}V^t \quad (\because V^tV = I)$

iii ) $\underline{x}^t\Sigma^{-1}\underline{x} = \underline{x}^t(VD_{1/\lambda}V^t)\underline{x} = 1$

$\Leftrightarrow \quad (\underline{x}^tV)D_{1/\lambda}(\underline{x}^tV)^t = 1$

let $V^t\underline{x} = \underline{y} = (y_1, \cdots, y_p)^t$ : pc transformation

$\Leftrightarrow \quad \underline{y}^tD_{1/\lambda}\underline{y} = 1$

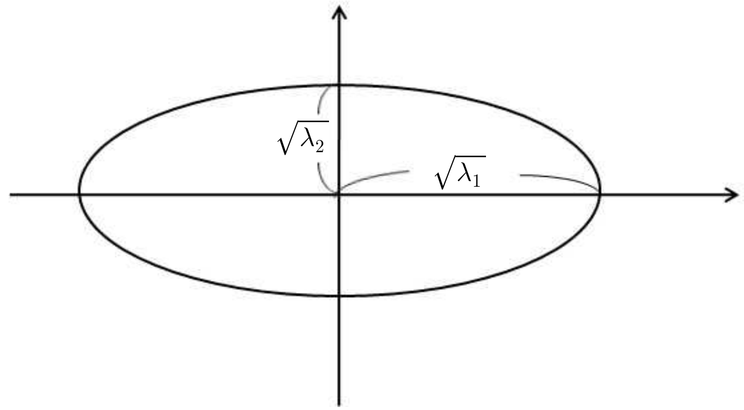
$$\Leftrightarrow \quad (y_1 \cdots y_p)\begin{pmatrix} \frac{1}{\lambda_1} & & \underline{0} \\ & \ddots & \\ \underline{0} & & \frac{1}{\lambda_p} \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} = 1$$

$\Leftrightarrow \quad \frac{y_1^2}{\lambda_1} + \cdots + \frac{y_p^2}{\lambda_p} = 1$ : The equation of a ellipsoid

- $p = 2$

$$\frac{y_1^2}{\lambda_1} + \frac{y_2^2}{\lambda_2} = 1$$

$$y_1 = \pm\sqrt{\lambda_1},\ \ y_2 = \pm\sqrt{\lambda_2}$$




- $p = 3$

$$\frac{y_1^2}{\lambda_1} + \frac{y_2^2}{\lambda_2} + \frac{y_3^2}{\lambda_3} = 1$$