# Multivariate Statistics (I)

## 3. Factor Analysis (FA)

# Contents

# 3.1 Comprehension of FA

- **Definition**

FA: technique for describing the covariance relationship among many variables in terms of a few *factors* which are underlying, but unobservable random quantities.

**History :**

- **K. Pearson and Charles Spearman provided beginnings of FA in the early 20th century.**
- Charles Spearman is known for being the one who coined the term factor analysis and actually used it to measure children's cognitive performance.
- Spearman, C. (1904). General intelligence objectively determined and measured, *American Journal of Psychology*, 15, 201–293.

## Charles Spearman

From Wikipedia, the free encyclopedia

**Charles Edward Spearman**, FRS (10 September 1863 - 17 September 1945) was an English psychologist known for work in statistics, as a pioneer of factor analysis, and for Spearman's rank correlation coefficient. He also did seminal work on models for human intelligence, including his theory that disparate cognitive test scores reflect a single general factor and coining the term *g* factor.

# 3.1 Introduction of FA - Process Steps for FA

- [STEP 1] Prepare a multivariate data matrix X.
- [STEP 2] Obtain a covariance matrix S (or a correlation matrix R).
- [STEP 3] PCFA is performed to obtain the first $m(\leq p)$ common factors of 70% or more of the goodness-of-fit, and the common factors are can be interpreted after the orthogonal transformation.
- [STEP 4] MLFA is performed and the common factors are obtained through the test and the common factors are interpreted after the orthogonal transformation.
- [STEP 5] Compare the tendency of the common factors and the factor scores obtained from [STEP3] and [STEP 4].
- [STEP 6] Repeat [Step 3] - [Step 5] while changing the number of common factors.
- [STEP 7] Consider factor scores as a new multivariate data reducing dimensionally .

# 3.2 Concept of common factor

- Spearman's study of 1904 : $n$ = 33 children of private elementary school

Classic French English Mathematics Discrimination of pitch Music

$R =$

| | 고전 | 프랑스어 | 영어 | 수학 | 음감 | 음악 |
|---|---|---|---|---|---|---|
| 고 전 | 1.00 | | | | | |
| 프랑스어 | 0.83 | 1.00 | | | | |
| 영 어 | 0.78 | 0.67 | 1.00 | | | |
| 수 학 | 0.70 | 0.67 | 0.64 | 1.00 | | |
| 음 감 | 0.66 | 0.65 | 0.54 | 0.45 | 1.00 | |
| 음 악 | 0.63 | 0.57 | 0.51 | 0.51 | 0.40 | 1.00 |

Factor loadings

General ability(intelligence) factor

$$x_j = \lambda_j f + e_j, \quad j = 1, \ldots, 6.$$

Specific factor

PCFA

$$\lambda_1 = -0.94, \ \lambda_2 = -0.89, \ \lambda_3 = -0.84, \ \lambda_4 = -0.80, \ \lambda_5 = -0.74, \ \lambda_6 = -0.72,$$

# 3.3 Factor model

Vector of specific factors

❖ Model with $m$ common factors $\quad \boldsymbol{x} = (x_1, \ldots, x_p)^t \sim \ (\boldsymbol{\mu}, \Sigma\ ), \quad \Sigma > 0$

$$x_j - \mu_j = \lambda_{j1} f_1 + \cdots + \lambda_{jm} f_m + e_j, \quad j = 1, \ldots, p \quad \Rightarrow \quad \boldsymbol{x} - \boldsymbol{\mu} = \Lambda \boldsymbol{f} + \boldsymbol{e}$$

Matrix of factor loadings

❖ Assumptions

$$\boldsymbol{f} \text{ and } \epsilon : \text{ independent} \Leftrightarrow \quad \mathrm{Cov}(\boldsymbol{\epsilon}, \boldsymbol{f}) = E(\boldsymbol{\epsilon} \boldsymbol{f}') = \mathbf{0}$$

$$E(\boldsymbol{f}) = \mathbf{0}, \ \mathrm{Cov}(\boldsymbol{f}) = E(\boldsymbol{f}\boldsymbol{f}') = \boldsymbol{I}$$

$$E(\boldsymbol{f}) = \mathbf{0}, \ \mathrm{Cov}(\boldsymbol{\epsilon}) = E(\boldsymbol{\epsilon}\boldsymbol{\epsilon}') = \Psi = diag(\psi_1, \ldots, \psi_p)$$

❖ Properties : Covariance Structure

1) $\Sigma = \Lambda \Lambda^t + \Psi$ : **Common factors decomposition**

- $var(x_j) = \sigma_{jj} = \lambda_{j1}^2 + \cdots + \lambda_{jm}^2 + \psi_j = h_j^2 + \psi_j$
- $cov(x_j, x_k) = \sigma_{jk} = \lambda_{j1}\lambda_{k1} + \cdots + \lambda_{jm}\lambda_{km}$
- $h_j^2 = \lambda_{j1}^2 + \cdots + \lambda_{jm}^2 = \sum_{k=1}^{m} \lambda_{jk}^2$ : $jth$ communality (sum of squared loading of the $x_j$)

2) $Cov(\boldsymbol{x}, \boldsymbol{f}) = \Lambda$

- $cov(x_j, f_k) = \lambda_{jk}$ : loadings of the $jth$ variable $x_j$ $on\ the\ kth\ factor$)

# 3.4 Estimation of factor model - PCFA

[Table 3.4.1] PCFA of S based on the PC method

[step 1] Data matrix : $X=[\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n]^t$, $\boldsymbol{x}_i=(x_{i1},\cdots,x_{ip})^t$, $i=1,\cdots,n$.

[step 2] Centred data matrix : $Y=HX$, $H=I-n^{-1}1_n 1_n^t$.

[step 3] Spectral decomposition :

$$\boxed{Y^tY/(n-1)=S=VDV^t=\sum_{k=1}^{p} l_k \boldsymbol{v}_k \boldsymbol{v}_k^t}$$

- $V=(\boldsymbol{v}_1,\cdots,\boldsymbol{v}_p)$ : Orthogonal matrix satisfying $V^tV=VV^t=I$
- $D=diag(l_1,\cdots,l_p)$ : A matrix of eigenvalues satisfying $l_1\geqq\cdots\geqq l_p>0$

[step 4] Proportion of total sample variance due to $j$th factor : $\dfrac{l_k}{\sum_{j=1}^{p} s_{jj}}\times 100$, $k=1,\ \ldots,\ m$

[step 5] Estimation of factor loading matrix: $\hat{\Lambda}=\hat{\Lambda}_y=\left[\sqrt{l_1}\,\boldsymbol{v}_1,\cdots,\sqrt{l_m}\,\boldsymbol{v}_m\right]$, $m<p$.

[step 6] Estimation of specific variance :

$$\hat{\Psi}=\hat{\Psi}_y=diag(\widehat{\psi}_{y1},\cdots,\widehat{\psi}_{yp}),\widehat{\psi}_{y_j}=s_{jj}-\sum_{k=1}^{m}\widehat{\lambda}_{y_{jk}}^2$$

[step 7] Residual matrix : $R_e=S-(\widehat{\Lambda}_y\widehat{\Lambda}_y^t+\hat{\Psi}_y)$

# 3.4 Estimation of factor model : PCFA

- **How do we select the number of factors m in PCM?**
  - ✓ Set $m$ equal to the number of eigenvalues of R greater than 1 or the number of positive eigenvalues of S (Rule of thumb= Kaiser(1960)'s rule)
  - ✓Percentage of variation accounted for by the $\boxed{\text{first } m \text{ eigenvalues}}$ are more than equal to about 70%, i.e.

1) $$\frac{\sum_{k=1}^{m} l_k}{\sum_{j=1}^{p} s_{jj}} \times 100 \geq 70\% \text{ for S}$$

2) $$\frac{\sum_{k=1}^{m} l_k}{p} \times 100 \geq 70\% \text{ for R}$$

  - ✓ Residual matrix: $R_e = S - (\hat{\Lambda}_y \hat{\Lambda}_y^t + \hat{\Psi}_y)$ vs. $R_e = R - (\hat{\Lambda}_z \hat{\Lambda}_z^t + \hat{\Psi}_z)$

The diagonal elements are zero and the other elements are small: $m$ factors model is appropriate !

# 3.4 Estimation of factor model : PCFA

## [Example 3.4.1] PCFA of KLPGA Data (klpgaa.txt)

- **[STEP 1] Prepare a multivariate data matrix X form [Data 1.3.2]**
- **[STEP 2] Obtain a covariance matrix S (or a correlation matrix R).**

$R =$

| | 평균퍼팅수 | 그린적중률 | 파세이브율 | 파브레이크율 | 평균타수 | 상금률 |
|---|---|---|---|---|---|---|
| 평균퍼팅수 | 1.000 | 0.128 | -0.376 | -0.440 | 0.444 | -0.407 |
| 그린적중률 | 0.128 | 1.000 | 0.759 | 0.731 | -0.800 | 0.641 |
| 파세이브율 | -0.376 | 0.759 | 1.000 | 0.717 | -0.937 | 0.736 |
| 파브레이크율 | -0.440 | 0.731 | 0.717 | 1.000 | -0.897 | 0.829 |
| 평균타수 | 0.444 | -0.800 | -0.937 | -0.897 | 1.000 | -0.829 |
| 상금률 | 0.407 | 0.641 | 0.736 | 0.829 | -0.829 | 1.000 |

- **[STEP 3] Spectral decomposition** – $R = VDV^t$ : $V = (v_1, \dots, v_p)$, $D = diag(l_1, \dots, l_p)$, $l_1 \geq \cdots \geq l_p > 0$.

Eigenvector : $V = (v_1,\ v_2,\ v_3,\ v_4,\ v_5,\ v_6)$ eigenvalue : $(l_1, \dots, l_6) = (4.31,\ 1.12,\ 0.33,\ 0.20,\ 0.03,\ 0.01)$

| $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
|---|---|---|---|---|---|
| -0.21 | 0.84 | -0.14 | -0.16 | 0.45 | 0.04 |
| 0.39 | 0.53 | 0.06 | 0.23 | -0.71 | -0.06 |
| 0.44 | 0.04 | 0.66 | -0.27 | 0.27 | -0.47 |
| 0.45 | -0.05 | -0.47 | 0.56 | 0.38 | -0.35 |
| -0.48 | 0.00 | -0.20 | -0.12 | -0.25 | -0.81 |
| 0.43 | -0.07 | -0.53 | -0.72 | -0.09 | 0.01 |

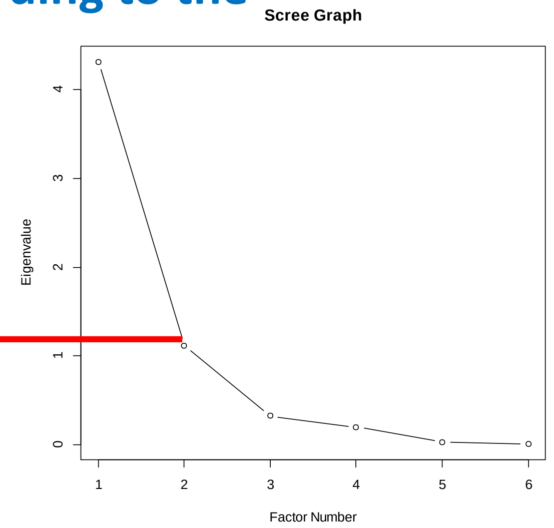
**Putting average**
**Green in regulation %**
**Par save %**
**Par break %**
**Scoring average**
**Prize rate**

# 3.4 Estimation of factor model : PCFA

- **[STEP 4] Proportion of total sample variance according to the number of factors**

$$l_1 = 4.31 \;,\; l_2 = 1.12 \;\longrightarrow\; \frac{4.31+1.12}{6} \times 100 = 90.48\%$$

$$m = 2 \;\longleftarrow$$




- **[STEP 5] Estimation of factor loading matrix**

$$\hat{\Lambda} = \hat{\Lambda}_z = \left[ \sqrt{l_1}\, \boldsymbol{v}_1, \ldots, \sqrt{l_m}\, \boldsymbol{v}_m \right],\ m < p.$$

$$- \ \hat{\Lambda} = \hat{\Lambda}_z = \left[ \sqrt{l_1}\, \boldsymbol{v}_1, \sqrt{l_2}\, \boldsymbol{v}_2 \right] = \left[ \sqrt{4.31}\, \boldsymbol{v}_1, \sqrt{1.12}\, \boldsymbol{v}_2 \right]$$

- **[STEP 6] Estimation of specific variance**

$$\hat{\Psi} = \hat{\Psi}_z = \ diag(\widehat{\psi}_{z_1}, \ldots, \widehat{\psi}_{z_p}),\ \widehat{\psi}_{z_j} = 1 - \sum_{k=1}^{m} \widehat{\lambda}_{z_{jk}}^2$$

# 3.4 Estimation of factor model : PCFA

- **[STEP 7] Residual matrix** $R_e = R - (\hat{\Lambda}_z \hat{\Lambda}_z^t + \hat{\Psi}_z)$

$R =$

| | 평균퍼팅수 | 그린적중률 | 파세이브율 | 파브레이크율 | 평균타수 | 상금률 |
|---|---|---|---|---|---|---|
| 평균퍼팅수 | 0.000 | -0.017 | -0.015 | 0.014 | 0.010 | 0.048 |
| 그린적중률 | -0.017 | 0.000 | -0.004 | 0.004 | 0.007 | -0.040 |
| 파세이브율 | -0.015 | -0.004 | 0.000 | -0.134 | -0.027 | -0.076 |
| 파브레이크율 | 0.014 | 0.004 | -0.134 | 0.000 | 0.034 | -0.009 |
| 평균타수 | 0.010 | 0.007 | -0.027 | 0.034 | 0.000 | 0.061 |
| 상금률 | 0.048 | -0.040 | -0.076 | -0.009 | 0.061 | 0.000 |

Results of PCFA

| | Factor loadings | | Communalities | Specific variances |
|---|---|---|---|---|
| variables | $\lambda_{jk} = \sqrt{l_k}\, v_{jk}$ $f_1$ | $f_2$ | $h_j^2 = \lambda_{j1}^2 + \lambda_{j2}^2$ | $\psi_j = 1 - h_j^2$ |
| Putting average | -0.436 | $\boxed{0.888}$ | 0.979 | 0.021 |
| Green in regulation % | 0.810 | 0.561 | 0.970 | 0.030 |
| Par save % | 0.913 | 0.042 | 0.836 | 0.164 |
| Par break % | 0.934 | -0.053 | 0.876 | 0.124 |
| Scoring average | -0.997 | 0.000 | 0.993 | 0.007 |
| Prize rate | 0.893 | -0.074 | 0.802 | 0.198 |
| eigenvalue | 4.31 | 1.12 | | |
| contribution rate | 71.83 | 18.65 | total contribution rate = 90.48% | |

# 3.4 Estimation of factor model - MLFA

**MLFA based on the Maximum Likelihood Method**

$$f(\boldsymbol{x}) = \frac{1}{(\sqrt{2\pi})^p |\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^t \Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\}$$

[step 1] Given $\boldsymbol{x} = (x_1, \cdots, x_p)^t \sim N_p(\boldsymbol{\mu}, \Sigma)$, $\Sigma > 0$ , consider the likelihood function

$$l(\boldsymbol{\mu}, \Sigma) = \log L(\boldsymbol{\mu}, \Sigma) = -\frac{n}{2}\log|2\pi\Sigma| - \frac{1}{2}\sum_{i=1}^{n}(\boldsymbol{x}_i - \boldsymbol{\mu})^t \Sigma^{-1}(\boldsymbol{x}_i - \boldsymbol{\mu})$$

$$S_n = \frac{n-1}{n} S$$

$$= -\frac{n}{2}\log|2\pi\Sigma| - \frac{n}{2}tr(\Sigma^{-1}S_n) - \frac{n}{2}(\bar{\boldsymbol{x}} - \boldsymbol{\mu})^t \Sigma^{-1}(\bar{\boldsymbol{x}} - \boldsymbol{\mu})$$

[step 2] $\hat{\boldsymbol{\mu}} = \bar{\boldsymbol{x}}$ ; $\boxed{\Sigma = \Lambda\Lambda^t + \Psi} \rightarrow \boxed{\rho = D_\sigma^{-1/2}\Sigma D_\sigma^{-1/2}}$

$$l(\hat{\boldsymbol{\mu}}, \Lambda, \Psi) = -\frac{n}{2}\left[\log|2\pi(\Lambda\Lambda^t + \Psi)| + tr((\Lambda\Lambda^t + \Psi)^{-1}S_n)\right]$$

[step 3]

$$\boxed{\max_{\Lambda,\Psi} l}$$

$$(\hat{\Psi}^{-1/2}S_n\hat{\Psi}^{-1/2})\hat{\Psi}^{-1/2}\hat{\Lambda} = \hat{\Psi}^{-1/2}\hat{\Lambda}(I + \hat{\Lambda}^t\hat{\Psi}^{-1}\hat{\Lambda})$$

$$\hat{\Psi} = diag(S_n - \hat{\Lambda}\hat{\Lambda}^t)$$

$$\hat{\Lambda}^t\hat{\Psi}^{-1}\hat{\Lambda} \;:\; \text{diagonal matrix}$$

$$\rightarrow \boxed{\hat{\Lambda} = \hat{\Lambda}_y, \quad \hat{\Psi}_y} \rightarrow \boxed{\hat{\Lambda}_z = D_{\hat{\sigma}}^{-1/2}\hat{\Lambda}_y, \quad \hat{\Psi}_z = D_{\hat{\sigma}}^{-1}\hat{\Psi}_y}$$

# 3.4 Estimation of factor model: MLFA

- **How do we select the number of factors m in MLM?**

✓ Goodness-of fit: $\dfrac{\sum_{k=1}^{m} l_k}{\sum_{j=1}^{p} s_{jj}} \times 100$ for S $\qquad \dfrac{\sum_{k=1}^{m} l_k}{p} \times 100$ for R

✓Test the hypotheses $H_0 : \Sigma = \Lambda\Lambda^t + \Psi$ with an appropriate *m* vs. $H_1 : \Sigma \neq \Lambda\Lambda^t + \Psi$

$$\left(n - \frac{2p + 4m + 11}{6}\right)\ln\left(\frac{|\hat{\Lambda}\hat{\Lambda}^t + \hat{\Psi}|}{|(n-1)S/n|}\right) \simeq \chi^2_{([(p-m)^2 - p - m]/2)}$$

: Bartlett's test statistic based on the chi-square approximation when n ∧ n-p are large

✓ ⟵ Likelihood Ratio Test : $-2\log\Lambda \sim \chi^2_{df}$

✓Residual matrix : $\boxed{\hat{R}_e = S - (\hat{\Lambda}_y\hat{\Lambda}_y^t + \hat{\Psi}_y)} \rightarrow \boxed{\hat{R}_e = R - (\hat{\Lambda}_z\hat{\Lambda}_z^t + \hat{\Psi}_z)}$

The diagonal elements are zero and the other elements are small: *m* factor model is appropriate !

# 3.4 Estimation of factor model: MLFA

## ❖ [Example 3.4.3] MLFA of KLPGA

**Comparison of MLFA and PCFA**

| variable | MLFA $\hat{\lambda}_{zjk}$ $f_1$ | MLFA $\hat{\lambda}_{zjk}$ $f_2$ | MLFA $\hat{\psi}_{zj} = 1 - \hat{h}_{zj}^2$ | PCFA $\lambda_{jk} = \sqrt{l_k} v_{jk}$ $f_1$ | PCFA $\lambda_{jk} = \sqrt{l_k} v_{jk}$ $f_2$ | PCFA $\psi_j = 1 - h_j^2$ |
|---|---|---|---|---|---|---|
| Putting average | -0.811 | 0.581 | 0.005 | -0.436 | 0.888 | 0.021 |
| Green in regulation % | 0.455 | 0.855 | 0.062 | 0.810 | 0.561 | 0.030 |
| Par save % | 0.805 | 0.476 | 0.124 | 0.913 | 0.042 | 0.164 |
| Par break % | 0.814 | 0.382 | 0.191 | 0.934 | -0.053 | 0.124 |
| Scoring average | -0.882 | -0.467 | 0.005 | -0.997 | 0.000 | 0.007 |
| Prize rate | 0.754 | 0.352 | 0.307 | 0.893 | -0.074 | 0.198 |
| Contribution rate | 58.70 | 29.70 | total contribution rate = 88.40% | 71.83 | 18.65 | total contribution rate = 90.48% |

$\hat{R}_e =$

| | 평균퍼팅수 | 그린적중률 | 파세이브율 | 파브레이크율 | 평균타수 | 상금률 |
|---|---|---|---|---|---|---|
| 평균퍼팅수 | 0.000 | 0.000 | 0.001 | -0.001 | 0.000 | 0.000 |
| 그린적중률 | 0.000 | 0.000 | -0.015 | 0.034 | 0.000 | -0.003 |
| 파세이브율 | 0.001 | -0.015 | 0.000 | -0.121 | -0.005 | -0.039 |
| 파브레이크율 | -0.001 | 0.034 | -0.121 | 0.000 | -0.001 | 0.080 |
| 평균타수 | 0.000 | 0.000 | -0.005 | -0.001 | 0.000 | 0.000 |
| 상금률 | 0.000 | -0.003 | -0.039 | 0.080 | 0.000 | 0.000 |

Residual matrix of PCFA

# 3.5 Factor rotation and factor loadings plot

- **Definition**
  - An orthogonal transformation of the factor loadings : $T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

    Clockwise Counterclockwise
- **Concepts**
  - $\hat{\Lambda}$ the $p$ x $m$ matrix of estimated factor loadings obtained by PCFA or MLFA
  - $\hat{\Lambda}^* = \hat{\Lambda} T$ : matrix of rotated loadings where $TT^t = T^t T = I$

=> The estimated factor common decomposition remains unchanged:

Conclusion

$$\Sigma \doteq \hat{\Lambda}\hat{\Lambda}^t + \hat{\Psi} = \hat{\Lambda}(TT^t)\hat{\Lambda}^t + \hat{\Psi} = \hat{\Lambda}^*\hat{\Lambda}^{*t} + \hat{\Psi}$$

**From a mathematical viewpoint, it is immaterial whether $\hat{\Lambda}$ or $\hat{\Lambda}^* = \hat{\Lambda} T$ is obtained.**

**Since the original loadings may not be readily interpretable, it is usual practice to rotate them until a simple structure is achieved.**

Question: How can you choose an orthogonal matrix T? Varimax

$$\max_T \quad \nu = \frac{1}{p}\sum_{k=1}^{m}\sum_{j=1}^{p}(d_{jk}^2 - \bar{d}_k)^2 = \frac{1}{p}\sum_{k=1}^{m}\left[\sum_{j=1}^{p} d_{jk}^4 - p\bar{d}_k^2\right]$$

$$d_{jk} = \hat{\lambda}_{jk}^* / \hat{h}_j \rightarrow \bar{d}_k = p^{-1}\sum_{j=1}^{p} d_{jk}^2$$

Standardizing

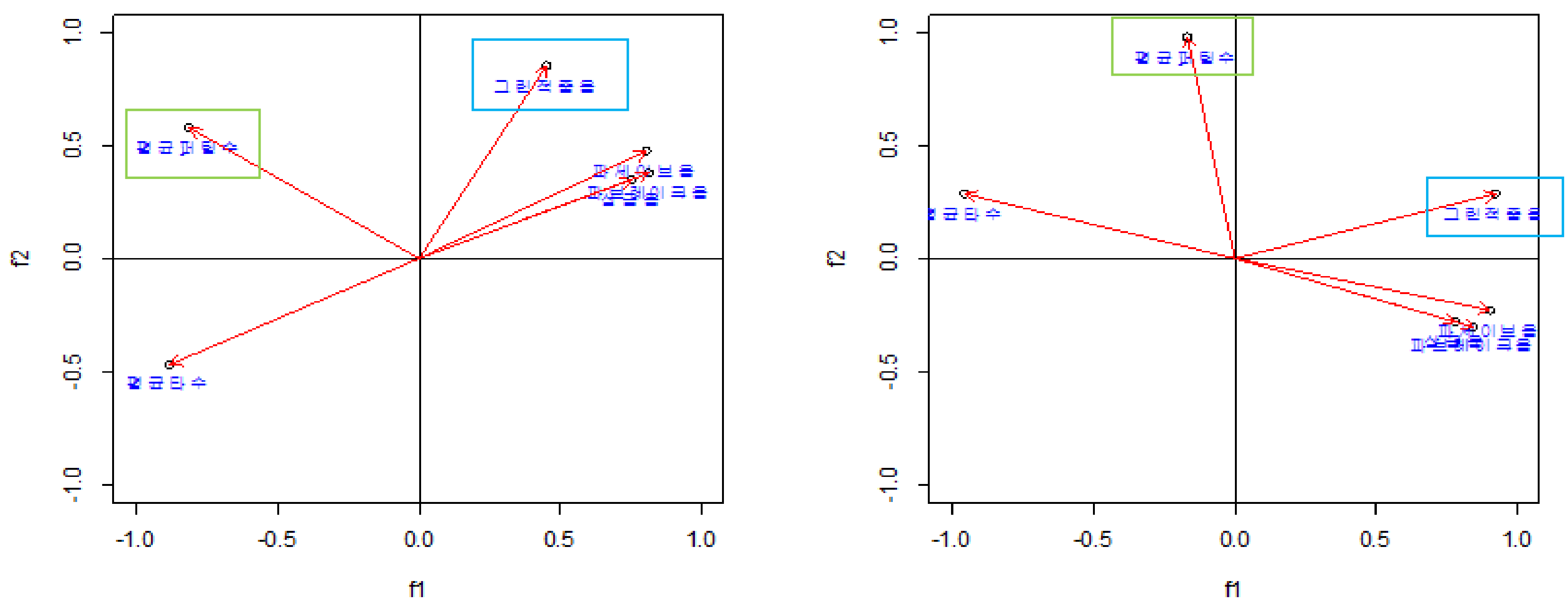
$$\hat{h}_j^2 = \hat{\lambda}_{j1}^2 + \cdots + \hat{\lambda}_{jm}^2$$

$$\hat{\Lambda}^* = \hat{\Lambda} T = (\hat{\lambda}_{jk}^*), \quad j = 1, \ldots, p, \quad k = 1, \ldots, m$$

# 3.5 Factor rotation and factor loadings plot

## [Example 3.5.1] Comparison of before and after the rotation in MLFA

| variable | Before rotation $\widehat{\Lambda}_z$ $f_1$ | Before rotation $\widehat{\Lambda}_z$ $f_2$ | After rotation $\widehat{\Lambda}_z^*$ $f_1$ | After rotation $\widehat{\Lambda}_z^*$ $f_2$ |
|---|---|---|---|---|
| Putting average | -0.811 | 0.581 | -0.168 | 0.983 |
| Green in regulation % | 0.455 | 0.855 | 0.925 | 0.288 |
| Par save % | 0.805 | 0.476 | 0.908 | -0.228 |
| Par break % | 0.814 | 0.382 | 0.848 | -0.301 |
| Scoring average | -0.882 | -0.467 | -0.955 | 0.288 |
| Prize rate | 0.754 | 0.352 | 0.784 | -0.280 |




(a) before rotation

Counterclockwise

(b) after rotation

# 3.6 Application of factor scores

Sum of the weighted squares of the errors

$$\min_{f_i} \boxed{e_i^t \Psi^{-1} e_i = (x_i - \mu - \Lambda f_i)^t \Psi^{-1} (x_i - \mu - \Lambda f_i)}$$

**Estimations of Factor Score**

- MLFA-WLSM: $\hat{f}_i^{ls} = (\hat{\Lambda}^t \hat{\Psi}^{-1} \hat{\Lambda})^{-1} \hat{\Lambda}^t \hat{\Psi}^{-1} (x_i - \bar{x}),\ i = 1, \ldots, n$
- MLFA-REGM: $\hat{f}_i^{re} = \hat{\Lambda}^t S^{-1} (x_i - \bar{x}),\ i = 1, \ldots, n$

  $\hat{f}_i^{re} = \hat{\Lambda}_z^t R^{-1} z_i,\ i = 1, \ldots, n$
- PCFA-LSM: $\hat{f}_i^{pc} = (\hat{\Lambda}^t \hat{\Lambda})^{-1} \hat{\Lambda}^t (x_i - \bar{x}),\ i = 1, \ldots, n$

$f \sim N_m(0,\ I).$

$x - \mu \sim N_p(0,\ \Sigma)$

$\longrightarrow\ f|x \sim N_m(\Lambda^t \Sigma^{-1}(x - \mu),\ I_m - \Lambda^t \Sigma^{-1} \Lambda) \longrightarrow \boxed{E(f|x) = \Lambda^t \Sigma^{-1}(x - \mu)}$

**R: Factor Scores**

pcfa=principal() ⇒ fpc = pcfa$scores

mlfa=factanal() ⇒ fml = mlfa$scores

library(psych)

# 3.6 Application of factor scores

❖ [Example 3.6.1] PCFA and MLFA of air-pollution data [Data 2.8.2] in LA

❖ **Comparison of varimax rotations of PCFA and MLFA : [R-code 3.6.1]**

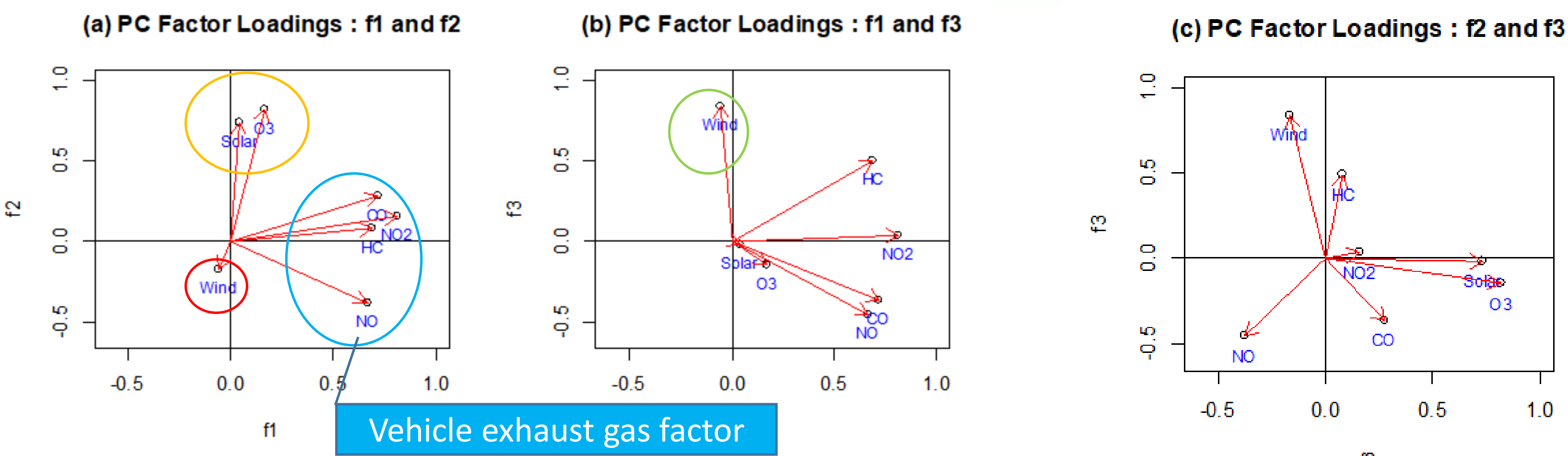
| | PCFA | | | | MLFA | | | |
|---|---|---|---|---|---|---|---|---|
| variable | $f_1$ | $f_2$ | $f_3$ | Specific distribution | $f_1$ | $f_2$ | $f_3$ | Specific distribution |
| Wind | -0.059 | −0.172 | 0.839 | 0.263 | 0.000 | -0.210 | -0.334 | 0.840 |
| Solar | 0.040 | 0.736 | -0.017 | 0.456 | 0.000 | 0.318 | 0.000 | 0.891 |
| CO | 0.718 | 0.278 | -0.364 | 0.275 | 0.487 | 0.318 | 0.507 | 0.405 |
| NO | 0.665 | −0.380 | -0.456 | 0.205 | 0.238 | -0.269 | 0.931 | 0.005 |
| NO2 | 0.810 | 0.156 | 0.034 | 0.319 | 0.989 | 0.000 | 0.000 | 0.005 |
| O3 | 0.167 | 0.820 | -0.148 | 0.278 | 0.000 | 0.987 | 0.124 | 0.005 |
| HC | 0.687 | 0.076 | 0.495 | 0.278 | 0.427 | 0.103 | 0.172 | 0.778 |
| Contribution rate | 33.38 | 19.80 | 17.20 | total contribution rate : 70.38% | 30.00 | 21.00 | 19.00 | total contribution rate : 70% |

Vehicle exhaust gas factor

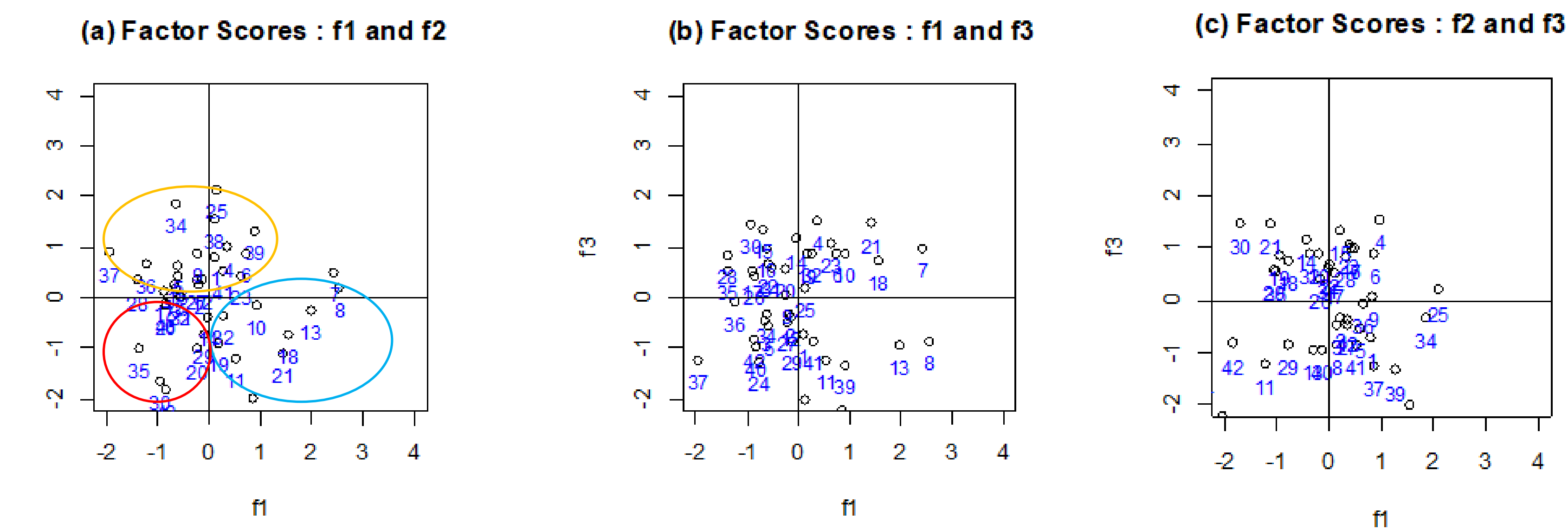
Residual Matrices of PCFA and MLFA: $\hat{R}_e$

| | PCFA | | | | | | | MLFA | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Wind | Solar | CO | NO | NO2 | O3 | HC | Wind | Solar | CO | NO | NO2 | O3 | HC |
| Wind | 0.000 | 0.042 | 0.202 | 0.087 | -0.064 | 0.021 | -0.205 | 0.000 | -0.032 | 0.072 | -0.001 | -0.001 | -0.001 | 0.261 |
| Solar | 0.042 | 0.000 | -0.056 | 0.172 | -0.031 | -0.294 | -0.023 | -0.032 | 0.000 | 0.043 | -0.001 | 0.000 | 0.000 | -0.018 |
| CO | 0.202 | -0.056 | 0.000 | -0.036 | -0.056 | 0.010 | -0.168 | 0.072 | 0.043 | 0.000 | 0.000 | 0.001 | 0.000 | -0.162 |
| NO | 0.087 | 0.172 | -0.036 | 0.000 | -0.166 | 0.000 | 0.033 | -0.001 | -0.001 | 0.000 | 0.000 | 0.000 | 0.000 | 0.002 |
| NO2 | -0.064 | -0.031 | -0.056 | -0.166 | 0.000 | -0.092 | -0.137 | -0.001 | 0.000 | 0.001 | 0.000 | 0.000 | 0.000 | 0.001 |
| O3 | 0.021 | -0.294 | 0.010 | 0.000 | -0.092 | 0.000 | 0.050 | -0.001 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.001 |
| HC | -0.205 | -0.023 | -0.168 | 0.033 | -0.137 | 0.050 | 0.000 | 0.261 | -0.018 | -0.162 | 0.002 | 0.001 | 0.001 | 0.000 |

# 3.6 Application of factor scores




[Figure 3.6.1] Factor loadings




[Figure 3.6.2] PCFA' s factor scores

# 3.6 Application of factor scores

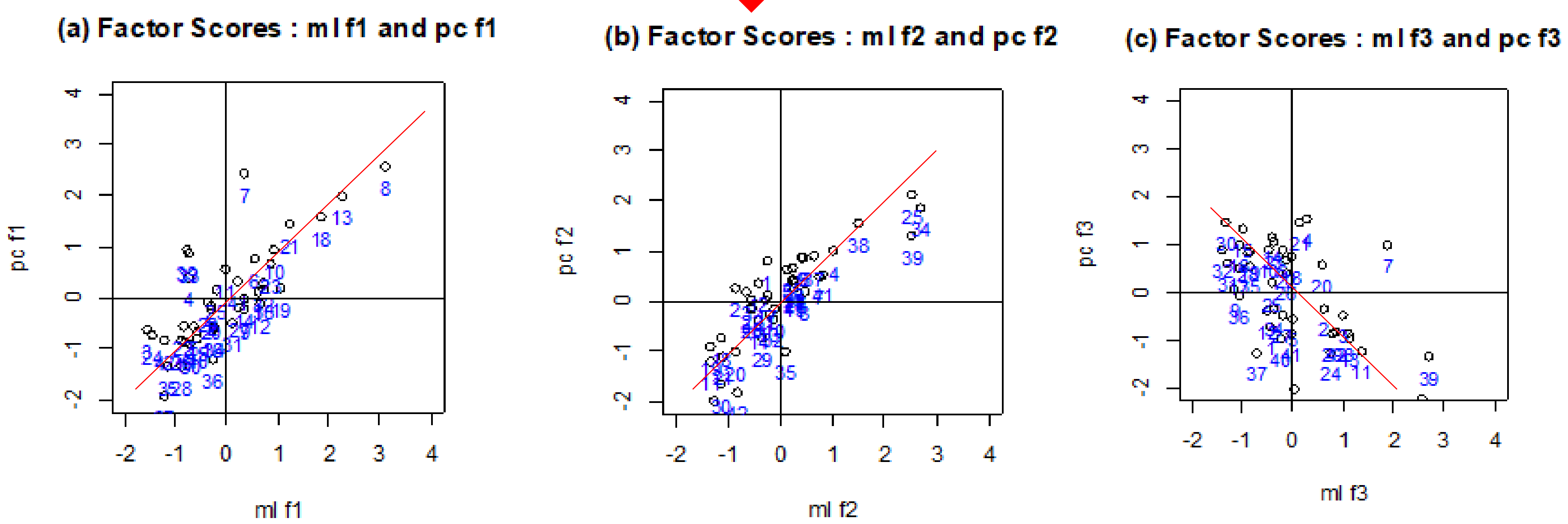
- 42 days LA air pollution

42 measurements on air-pollution variables recorded at 12:00 noon in the LA area on different days.

| | $X$: Law | | | | | | | $Z$: Standardized | | | | | | | $F$: Factor scores | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Wind | Solar | CO | NO | NO2 | O3 | HC | Wind | Solar | CO | NO | NO2 | O3 | HC | $f_1$ | $f_2$ | $f_3$ |
| 1 | 8 | 98 | 7 | 2 | 12 | 8 | 2 | 0.316 | 1.393 | 1.988 | -0.175 | 0.579 | -0.252 | -1.583 | 0.116 | 0.800 | -0.737 |
| 2 | 7 | 107 | 4 | 3 | 9 | 5 | 3 | -0.316 | 1.912 | -0.444 | 0.744 | -0.311 | -0.791 | -0.138 | -0.175 | 0.230 | -0.341 |
| 3 | 7 | 103 | 4 | 3 | 5 | 6 | 3 | -0.316 | 1.681 | -0.444 | 0.744 | -1.497 | -0.612 | -0.138 | -0.629 | 0.167 | -0.492 |
| 4 | 10 | 88 | 5 | 2 | 8 | 15 | 4 | 1.581 | 0.816 | 0.367 | -0.175 | -0.607 | 1.005 | 1.308 | 0.367 | 0.978 | 1.521 |
| 5 | 6 | 91 | 4 | 2 | 8 | 10 | 3 | -0.949 | 0.989 | -0.444 | -0.175 | -0.607 | 0.107 | -0.138 | -0.587 | 0.603 | -0.563 |
| 6 | 8 | 90 | 5 | 2 | 12 | 12 | 4 | 0.316 | 0.931 | 0.367 | -0.175 | 0.579 | 0.466 | 1.308 | 0.749 | 0.860 | 0.867 |
| 7 | 9 | 84 | 7 | 4 | 12 | 15 | 5 | 0.949 | 0.585 | 1.988 | 1.664 | 0.579 | 1.005 | 2.754 | 2.429 | 0.492 | 0.975 |
| 8 | 5 | 72 | 6 | 4 | 21 | 14 | 4 | -1.581 | -0.107 | 1.177 | 1.664 | 3.249 | 0.826 | 1.308 | 2.558 | 0.172 | -0.877 |
| 9 | 7 | 82 | 5 | 1 | 11 | 11 | 3 | -0.316 | 0.470 | 0.367 | -1.095 | 0.283 | 0.287 | -0.138 | -0.230 | 0.845 | 0.050 |
| 10 | 8 | 64 | 5 | 2 | 13 | 9 | 4 | 0.316 | -0.569 | 0.367 | -0.175 | 0.876 | -0.073 | 1.308 | 0.940 | -0.181 | 0.865 |
| | | | ............... | | | | | | | ............... | | | | | | ............... | |
| 38 | 5 | 86 | 7 | 2 | 13 | 18 | 2 | -1.581 | 0.700 | 1.988 | -0.175 | 0.876 | 1.544 | -1.583 | 0.137 | 1.548 | -2.038 |
| 39 | 7 | 79 | 7 | 4 | 9 | 25 | 3 | -0.316 | 0.297 | 1.988 | 1.664 | -0.311 | 2.802 | -0.138 | 0.909 | 1.293 | -1.362 |
| 40 | 7 | 79 | 5 | 2 | 8 | 6 | 2 | -0.316 | 0.297 | 0.367 | -0.175 | -0.607 | -0.612 | -1.583 | -0.817 | -0.126 | -0.976 |
| 41 | 6 | 68 | 6 | 2 | 11 | 14 | 3 | -0.949 | -0.338 | 1.177 | -0.175 | 0.283 | 0.826 | -0.138 | 0.306 | 0.524 | -0.877 |
| 42 | 8 | 40 | 4 | 3 | 6 | 5 | 2 | 0.316 | -1.953 | -0.444 | 0.744 | -1.201 | -0.791 | -1.583 | -0.845 | -1.830 | -0.849 |

# 3.6 Application of factor scores

Similar Pattern fort Factor Scores of PCFA and MLFA




[Figure 3.6.3] Factor Scores PCFA and MLFA after rotation

# 3.7 Visualizations of FA

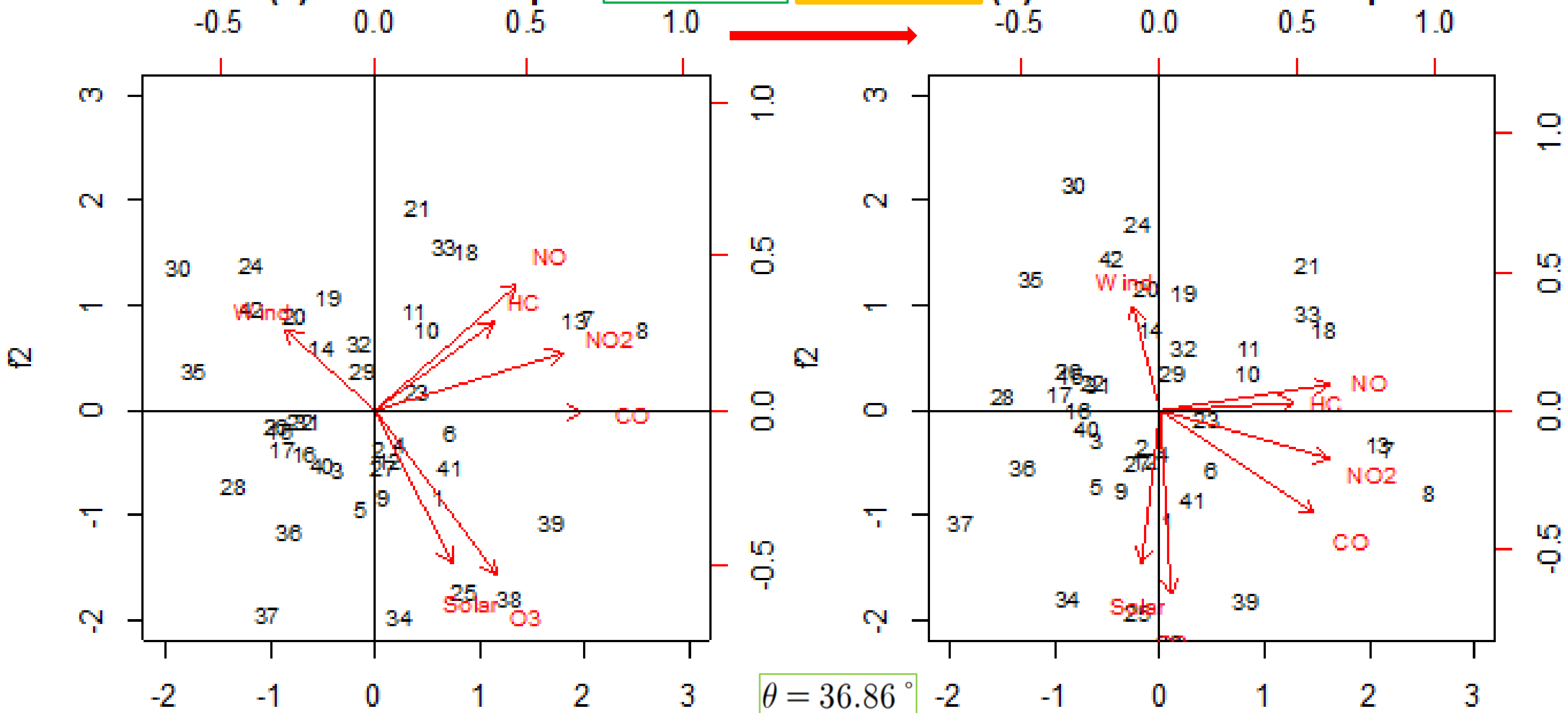
❖ [Example 3.7.1]Air pollution- FA Biplot

$$Y = U \Lambda V^t = \sum_{k=1}^{p} \lambda_k \boldsymbol{u}_k \boldsymbol{v}_k^t$$

$$\hat{\Lambda}_{(m)} = (n-1)^{-1/2} V_{(m)} \Lambda_{(m)} = (n-1)^{-1/2} [\lambda_1 \boldsymbol{v}_1, \ \dots, \ \lambda_m \boldsymbol{v}_m]$$

$$F_{(m)} = (n-1)^{1/2} U_{(m)} = (n-1)^{1/2} (\boldsymbol{u}_1, \ \dots, \ \boldsymbol{u}_m)$$

(a) Unrotated Biplot $\hat{\Lambda}^* = \hat{\Lambda} T$ $F^* = FT$ (b) Varimax Rotated Biplot




$\theta = 36.86^\circ$

$$T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0.837 & -0.547 \\ 0.547 & 0.837 \end{bmatrix}$$

# 3.8 R for FA : Practice Time

**R-Code:**

| FA(PCFA , MLFA) and FA Biplot | |
|---|---|
| library(psych), principal() | PCFA |
| library(psych), factanal() | MLFA |
| principal(, rotation="varimax")<br>factanal(, rotation="varimax") | Varimax Rotation |
| FA | Klpga-PCFAsteps-scree.R<br>Klpga-MLFAfactanal.R<br>Klpga-MLFAvarimax.R<br>airpollution-PCMLFAvarimax-scores.R |
| FA Biplot | airpollution-PCFAbiplot.R |

# 3.8 R for FA : Practice Time

R-code list of Chapter 3 Factor Analysis

| | | |
|---|---|---|
| spearman-PCFA.R | [R-코드 3.2.1] | 스피어만의 여섯 과목 성적의 PCFA |
| klpga-PCFAsteps-scree.R | [R-코드 3.4.1] | KLPGA 선수 성적의 PCFA |
| 5subjects-PCFAsteps.R | [R-코드 3.4.2] | 두 가지 시험성적의 PCFA |
| klpga-MLFAfactanal.R | [R-코드 3.4.3] | KLPGA 선수 성적의 MLFA |
| 5subjects-MLFAfactanal.R | [R-코드 3.4.4] | 두 가지 시험성적의 MLFA |
| klpga-MLFAvarimax.R | [R-코드 3.5.1] | KLPGA 선수 성적에 대한 MLFA의 Varimax 회전 후 인자적재그림 |
| 5subjects-MLFAvarimax.R | [R-코드 3.5.2] | 두 가지 시험성적에 대한 MLFA의 Varimax 회전 전과 후의 인자적재그림 |
| airpollution-PCMLFAvarimax-scores.R | [R-코드 3.6.1] | LA시 대기오염 자료의 PCMA와 MLFA의 실행과 비교 |
| airpollution-PCFAbiplot.R | [R-코드 3.7.1] | LA시 대기오염 자료의 PCFA의 회전 전과 후의 인자적재와 인자점수 행렬도 |

# 3.8 R for FA : Practice Time

[R-code 3.4.1] klpga-PCFAsteps-scree.R in [Example 3.4.1]

```
# PCFA Steps for KLPGA
#[Step 1] Data Matrix X
Data1.3.2<-read.table("klpga.txt", header=T)
X=Data1.3.2
rownames<-rownames(X)
p=ncol(X)
 #[Step 2] Covariance Matrix S(or Correlation Matix R)
 R=round(cor(X),3)
 R

 #[Step 3] Spectral Decomposition
 eigen.R=eigen(R)
 round(eigen.R$values, 2) # Eigenvalues
 V=round(eigen.R$vectors, 2) # Eigenvectors

 #[Step 4] Number of factors : m
 gof=eigen.R$values/p*100 # Goodness-of fit
 round(gof, 3)
 plot(eigen.R$values, type="b", main="Scree Graph",
              xlab="Factor Number", ylab="Eigenvalue")
```

```
#[Step 5] Factor Loadings and Communality
V2=V[,1:2]
L=V2%*%diag(sqrt(eigen.R$values[1:2]))
round(L, 3)
round(diag(L%*%t(L)), 3)

#[Step 6] Specific Variance : Psi
Psi=diag(R-L%*%t(L))
round(Psi, 3)

#[Step 7] Residual Matrix
Rm = R-(L%*%t(L) + diag(Psi))
round(Rm, 3)

# PCFA using the principal()
library(psych)
pcfa<-principal(R, nfactors=2, rotate="none")
pcfa
round(pcfa$values, 2)
gof=pcfa$values/p*100 # Goodness-of fit
round(gof, 3)
round(pcfa$residual, 2)
```

# 3.8 R for FA : Practice Time

[R-code 3.4.3] klpga-MLFAfactanal.R in [Example 3.4.3]

```
# MLFA Steps for KLPGA
# Data Matrix X
Data1.3.2<-read.table("klpga.txt", header=T)
X=Data1.3.2
rownames<-rownames(X)
p=ncol(X)
Z<-scale(X, scale=T)

# Covariance Matrix S(or Correlation Matix R)
R=round(cor(X),3)
R

# ML Estimation using the factanal( )
library(psych)
mlfa<-factanal(Z, factors = 2, rotation="none")
mlfa

# Residual Matrix
L=mlfa$loading[, 1:2]
Psi=mlfa$uniquenesses
Rm = R-(L%*%t(L) + diag(Psi))
round(Rm, 3)
```

# 3.8 R for FA : Practice Time

[R-code 3.5.1]  klpga-MLFAvarimax.R in [Example 3.5.1]

```
# MLFA  : None and Varimax Rotation for KLPGA
# Data Matrix X
Data1.3.2<-read.table("klpga.txt", header=T)
X=Data1.3.2
rownames<-rownames(X)
p=ncol(X)
# Covariance Matrix S(or Correlation Matix R)
R=round(cor(X),3)
R

# ML Estimation using the factanal( ): None
library(psych)
mlfa<-factanal(covmat=R, factors = 2, rotation="none" )
mlfa

# Residual Matrix
L=mlfa$loading[, 1:2]
Psi=mlfa$uniquenesses
Rm = R-(L%*%t(L) + diag(Psi))
round(Rm, 3)
```

```
par(mfrow=c(1,2))
# Factor Loadings Plot : None
lim<-range(pretty(L))
plot(L[,1], L[,2],main="Plot of Factor Loadings : None",  xlab="f1", ylab="f2",
                         xlim=lim, ylim=lim)
text(L[,1], L[, 2], labels=rownames(L), cex=0.8, col="blue", pos=1)
abline(v=0, h=0)
arrows(0,0, L[,1], L[, 2], col=2, code=2, length=0.1)

# ML Estimation using the factanal( ): Varimax
library(psych)
mlfa<-factanal(covmat=R, factors = 2,  rotation="varimax" ) # rotation="none"
mlfa

# Residual Matrix
L=mlfa$loading[, 1:2]
```

$\boxed{\hat{\Lambda}^* = \hat{\Lambda} T}$

```
L
Psi=mlfa$uniquenesses
Rm = R-(L%*%t(L) + diag(Psi))
round(Rm, 3)

# Factor Loadings Plot : Varimax
lim<-range(pretty(L))
plot(L[,1], L[,2],main="Plot of Factor Loadings : Varimax ",  xlab="f1", ylab="f2",
                         xlim=lim, ylim=lim)
text(L[,1], L[, 2], labels=rownames(L), cex=0.8, col="blue", pos=1)
abline(v=0, h=0)
arrows(0,0, L[,1], L[, 2], col=2, code=2, length=0.1)
```

# 3.8 R for FA : Practice Time

[R-code 3.7.1] airpollution-PCFAbiplot.R in [Example 3.7.1]

```
Data2.8.2<-read.table("airpollution.txt", header=T)
X=Data2.8.2
rownames(X)
colnames(X)
p=ncol(X)
n=nrow(X)
Z<-scale(X, scale=T)

# Biplot based on the Singular Value Decomposition
svd.Z <- svd(Z)
U <- svd.Z$u
V <- svd.Z$v
D <- diag(svd.Z$d)
F <- (sqrt(n-1)*U)[,1:2]  # Factor Scores Matrix : F
L <- (sqrt(1/(n-1))*V%*%D)[,1:2] # Factor Loadings Matrix : Lambda
C<- rbind(F, L)
rownames(F)<-rownames(X)
rownames(L)<-colnames(X)

# Godness-of-fit
eig <- (svd.Z$d)^2
per <- eig/sum(eig)*100
gof <- sum(per[1:2])
per
gof
```

```
# Biplot: Joint Plot of Factor Loadings and Scores
par(mfrow=c(1,2))
par(pty="s")
lim1 <- range(pretty(L))
lim2 <- range(pretty(F))
biplot(F,L, xlab="f1",ylab="f2", main=" (a)Unrotated Biplot", xlim=lim2, ylim=lim2,
cex=0.8,pch=16)

abline(v=0,h=0)

# Varimax Rotated Biplot: Joint Plot of Rotated Factor Loadings and Scores
varimax<-varimax(L)
Lt = varimax$loadings
T=varimax$rotmat
T
Ft= F%*%T
biplot(Ft,Lt, xlab="f1",ylab="f2", main=" Varimax Rotated Biplot",
                xlim=lim2,ylim=lim2,cex=0.8,pch=16)
abline(v=0,h=0)
```